Running head: USING RICH PROBLEMS FOR DIFFERENTIATED INSTRUCTION

Using Rich Problems for Differentiated Instruction

Eric Hsu, Judy Kysh, and Diane Resek

San Francisco State University

Abstract

Working in well-facilitated small groups on rich problems that are accessible puts students in the position of differentiating the content, processes, and product of their own work. When students are empowered to make natural choices as they work on rich problems together, there are almost always surprises for teachers and often for the students themselves. One of the most important surprises is who comes up with interesting ideas; it is not always who a teacher would have predicted. In this article we discuss what makes a problem rich enough to allow facilitation of this self-differentiated student work.

Using Rich Problems for Differentiated Instruction

Even when classes are arranged so that students of the same perceived ability are in the same classroom, teachers find their students have different strengths, work at different paces, and need different types of assistance. How can teachers respond to the differing needs of 25-30 students at the same time?

Tomlinson (1999, p.48) identifies three major components of instruction that can be differentiated: content covered, process experienced, or products expected. In many cases the teacher will identify one or more of these components, and then using his or her best judgment, and predetermine which approach each student will use. On the other hand, some teachers will provide alternative assignments and allow students to choose. In the approach we will describe, students all work on the same rich problem, and the teacher through listening, questioning, and responding to students as they work together in small groups, encourages them to explore new but accessible content, build on the processes they have started using, and come up with a product that extends their previous knowledge. We will use several examples as a basis for discussing criteria for selecting problems that can support this approach.

As co-directors of the REvitalizing ALgebra (REAL) project, a four-year NSF Math/Science Partnership, we worked with 20 secondary teachers across five school districts in a joint effort to improve the teaching and learning of algebra for all students, particularly minority students. We worked with two groups of teachers, starting in successive years. For each of the two cohorts, we met three hours a week during the first year and daily for three weeks during one summer. The following year the grant funding supported daily released time from teaching so each group of leaders and their department colleagues had time to work together on the activities their leaders had selected from our workshops, to plan richer algebra lessons together, and to observe these lessons in each others' heterogeneously organized middle and high school algebra classrooms.

What Does Classwork Around Rich Problems Look Like?

Some people find it counterintuitive to use bigger, harder problems to make math accessible to more students. It helps to have an image of how effective classwork on rich problems would look.

- Students are placed into groups that are purposefully not homogenous. We commonly group students using some public random process, such as counting off, so it is clear there is no pre-grouping by perceived strength.
- All students are given the same initial problem to work on in these non-homogenous groups. Groups who think they have finished early are asked to consider alternative methods of solution, further generalization, or other extension questions.
- Groups are expected to be responsible for the respectful learning of all members.[1]

[1] There is not space to treat all the possibilities and subtleties of facilitation techniques in this article. Some good resources: Cohen (1994) on complex instruction; Webb et al (2001) and Weissglass (1976) on group work; and Hsu et al (2007) on Emerging Scholars Program workshops.

- Students need to communicate their thinking so others can understand and build on it. Usually students are expected to share their work with the class, either in informal whole class discussion or as formal presentations, which sometimes include prepared overheads or posters.
- Students are expected to struggle with the problem, and are expected to negotiate and argue their different ideas.[2]
- Good mathematical argument and explanation are emphasized as goals and are necessary to the reporting process.
- Creative approaches are encouraged and analyzed, even if they don't lead directly to a solution.

In an effective class,

- Students will wrestle with **content** that is challenging but accessible to them. They will argue with other students about math they don't understand and try to persuade others of the math they do understand or partially understand.
- Students will begin with the same rich problem, but students who solve it quickly will work on questions that will deepen their understanding, and students who have difficulty will receive hints and productive probing questions from the teacher. In this way students

[2] This can be enforced through a strict facilitator policy of only taking questions from the whole group, and by asking randomly selected individuals to report results for the whole group

experience the **process** most appropriate for them for that task. Note that for truly rich problems, a student who takes the lead on one problem may have difficulty with another.

- Students will produce **products** that represent their different thinking. A sufficiently rich task will have a range of legitimate possible methods of solution with different levels of difficulty or abstraction. In addition to valuing the final product of a presentation, a teacher will learn a lot about student thinking and learning by valuing the argumentation and reasoning between students throughout the process, as many valuable ideas can arise that may not be apparent in the final presentation.

This approach comes with some major challenges. First, as one can imagine, it takes a special richness of problem to drive such a complex classroom, and these are sometimes difficult to find. The majority of texts lack interesting problems, or, if they are present, they are tucked away in the backs of chapters and require major restructuring. Second, once such problems are found, it takes skill to harness the power of rich problems through productive facilitation of the class.

It is hard to fully grasp these ideas through abstract discussion, so we have selected several specific examples of rich problems and provided analyses of what makes them rich and how they might be used.

What Makes A Problem “Rich” Enough For Differentiated Instruction?

The purpose of a rich problem is to provoke mathematical curiosity, struggle, conversation, and insight into important mathematical topics. Unfortunately, there is no way to guarantee such a result for every audience; however, we have developed some guidelines for choosing problems. In order to describe criteria for rich problems, we will analyze three problems that we have used successfully with groups of students from pre-algebra to calculus.

Problem 1. How many 1 foot by 1 foot square tiles are needed to form a border inside a square space measuring 10 feet by 10 feet? Find a way to solve this without counting every tile (from Alper, et al., 2003a).

Students are encouraged to find and share multiple ways of solving the problem, and then asked to solve the general problem for the border of an N by N square. The reader should stop and solve this problem in at least two ways.

Five key aspects of rich problems

1.) The "mysterious" part of the problem is mathematical. This may sound like a trivial observation, but a large number of school tasks either have no mystery (e.g. practice or application of some algorithm) or the mystery is non-mathematical (e.g. exercises where each solution reveals a clue to a puzzle, so the mystery is in the puzzle and not in the mathematics).

2.) The problem has very little visible scaffolding. By this we mean there is a situation described and students are asked to solve it without following a directed sequence of steps or a specific method. Presentation of the problem may be followed by a class discussion for clarification of what the problem is asking. For groups that are struggling, the facilitator will have prepared a structured set of "pocket questions," including such questions as, "Could you draw a picture?" or "Show me how you came up with that?" and "Could you break the border into parts?" or "How many tiles are in the different parts of the border?" When the initial task has very little upfront structure, the facilitator can introduce different levels of supporting questions as different groups can benefit from them. For example, in Clarke (1997) the authors propose a Continuum of Mathematical Tasks that, roughly speaking, run from closed and high-structure to open and low-structure. Also some texts provide further guidance questions to be used by some groups and teacher notes containing suggested "pocket questions" (Dietiker et al, 2005 and Alper et al, 2003a).

.

The problem's lack of upfront structure does not mean that the class session is unstructured. On the contrary, the teacher must be very clear about what students are expected to take away from the lesson. A plan for a suitable conclusion to the group work to assure that most students understand the mathematical point is essential. Closure activities might include selected presentations by some groups that highlight different aspects of the activity, a whole group discussion and summary, or posters from each group that summarize their findings.

3.) There are many ways to do the problem. In one sense, there is only one correct answer for this problem, 36 or 4s - 4. Yet, in this case, the answer is the least interesting part of the task. There are at least four common ways to solve it. The problem itself suggests drawing a diagram, and once that is drawn, students see legitimate and different ways of decomposing the diagram into pieces. Because the problem does have a brute-force counting solution, students can easily check their solutions for themselves.

4.) Students of different skill levels can learn from this activity. This is related to (3) but emphasizes the fact that the different legitimate approaches to the problem are accessible to students with very different backgrounds. Students who understand variables at different levels can benefit. Students who never realized how variables can be used to generalize arithmetic, can gain a beginning understanding, while others who are able to use variables to generalize may not have realized how geometric pictures can be represented in different, but equivalent ways. The fact that 4s –4 can be seen as four sides minus the corners could be a new way of looking at symbols for such students. Finally, students can be challenged to find several ways of representing a solution and gain new insight into equivalence of algebraic expressions.

5.) The problem has natural extensions. In addition to being asked to come up with several alternative ways to write expressions for the solution of the border problem, groups could work on the following pattern. After counting the bricks in stages 2 and 3 of the growth pattern shown in Figure 1, their task would be to find a formula for the number of bricks needed at the nth stage.

(insert Figure 1 here)

We will analyze a second example, emphasizing some additional key aspects. Again, we encourage the reader work on this problem before reading what follows.

<u>Problem 2</u>. Which positive integers can be written as the difference of two squared integers? For example, 17 = $9^2 - 8^2$ (from Sallee et al, 1991)**.**

Students given this problem would be pushed to explore and create many examples and then encouraged to establish some partial results that could be shared with the whole class. The facilitator would use some judgment in ordering "work in progress" class presentations, probably delaying solutions with more generality.

Three Additional Key Aspects of Rich Problems:

6.) <u>The problem is hard</u>. Problem 2 is a harder problem than Problem 1 in that no one short of a number theorist is going to see a complete answer right away. It is likely that a group will need to engage in considerable discussion to find a productive approach, and, as everyone will have trouble, there is no shame or loss of status for not knowing how to solve the problem right away. While the groups are working, the teacher must be prepared with questions to encourage those who are becoming too frustrated and/or unproductive.

7.) <u>The problem encourages getting your hands dirty with data</u>. There are a number of common ways of approaching this problem. Some will approach the problem algebraically looking at formulas such as $a^2 - b^2$ or $(a+n)^2 - a^2$. However, it is very difficult to find a complete description of the possible numbers from the algebra alone. No group we have worked with (including teachers, undergraduate math majors, and graduate students in mathematics) has succeeded with a purely algebraic approach.

Groups are going to need to try out some numbers and start collecting data, in itself a non-trivial task. Once a pattern emerges from the data, algebra can be used to justify the apparent pattern. In practice, many of the higher-status (in the sense of students strong at symbolic manipulation and using standard algorithms) students are going to try an algebraic approach, and many lower-status students will try to collect data. There is something very leveling about problems that require data collection. “Stronger” students will often rush ahead and get stuck in the algebra while “weaker” students will get partial or even full results. At some point the lower-status students will have the pleasure of explaining their work to the higher-status students. Sometimes a few students will go ahead and justify the result using algebra, or sometimes there will be a group effort. But in either case, the data-collecting students will have the pride of knowing their work was essential to finding a solution.

8.) The problem has interesting partial solutions. Many students find patterns, such as every odd number. This partial result has some nice explanations. Students usually reason that every odd number is the difference of successive squares either through algebra (writing an odd number as $2n + 1$ for some n, and then $(n+1)^2 – n^2 = n^2 +2n + 1 – n^2 = 2n +1$), or through elegant diagrams. Other people notice that the numbers that can't be written as the difference of two squares are separated by 4, or they notice that the multiples of 4 are possible, with or without proof. Thus, students are able to work on this rich problem at many levels and, at the same time, learn some algebra.

A lot can be learned without completing all aspects of the problem. Many students might find the patterns, but not the proof. Even without proofs, the different methods of collecting and organizing the data are in themselves interesting. Students sometimes can prove that odd numbers and multiples of 4 are possible, usually by examining $(n+2)^2 - n^2$, but cannot prove that other numbers are impossible. A complete answer would require some kind of argument to the effect, that, if

$x^2 – y^2 = (x – y) (x + y)$ is even, then by examining cases, both factors have to be even, and hence their product must have 4 as a factor.

While the first two examples are problems that emphasize ideas of variable, mathematical representation, exploration and proof, the technique of using rich problems is not limited to those areas of the curriculum. The results of the students' investigations can provide the teacher with formative assessment regarding their prior knowledge as well as their facility with creating new connections. The outcomes of this work will allow the teacher to design avenues for further investigation that move the students toward the overarching goal of the unit.

Often the introduction to a new unit of instruction can be turned into a rich problem. Instead of presenting new information while students take notes, teachers can pose a general question for student groups to investigate and report their findings. The teacher should be prepared with questions to push each group's investigation. During the summary activity, the teacher is able to fill in the gaps when students are more likely to have an interest in the subject since they have some ownership.

An example of this strategy is taken from the second year of Interactive Mathematics Program (Alper, et al, 2003b) .

<u>Problem 3</u>. In this activity, you will investigate linear functions and straight-line graphs. Here are some questions to explore.

How do you change the equation in order to change the "slant" of its graph?

How do you change the equation in order to shift the whole graph up or down?

When do two linear functions give parallel lines (lines that never meet)? Why?

What linear functions give horizontal lines? Why?

When do two linear functions give lines that are mirror images of each other with the y-axis as the mirror? Why?

When do two linear functions give perpendicular lines Why?

Use your graphing calculator and keep track of what you try. Do not feel limited by these questions -- let your imagination soar! Keep track of any other interesting questions you think of, even if you can't answer them.

9.) <u>This problem allows the teacher to find out what students know, build on it, and use the information to plan future lessons .</u> If none of the students is familiar with the material, all should be on equal footing with suggestions about what to try and in making observations. In most cases some students know some of the material, and the investigations will inform teachers of who knows what. In the process of the investigation other students will be learning new material, but the investigation is non-trivial enough that each group can find questions where no member is sure of an answer. For example, while some are learning about the relationship of the slope and y-intercept of a line to its equation, others will explore the relationship between the equations of perpendicular lines.

10.) <u>This problem is open enough to provide opportunities for students to be ingenious.</u> If teachers choose their questions well, they give students opportunities to come up with some of the math they were planning to tell them about, so everyone has a stake in what is coming next.

What Are Some Examples of Problems That Seem Rich But Are Not?

Above we have listed ten Key Aspects behind Rich Problems. Not every rich problem will have all the listed aspects, but they will satisfy many of them. In contrast, we include three brief examples of problems that appear rich, but are not when tried in the classroom. This doesn't mean they are bad problems, nor that they lack richness for every possible audience. For instance, many mundane problems become rich when framed as explorations for students who have not been exposed to the material before. Nevertheless, for the most commonly chosen audiences of these problems, they are less rich than the previous examples. You may want to try these problems before reading our discussion.

Problem 4. Find the secret word using the following clues. Solve each equation for a value for x which corresponds to a letter using the code A=1, B-2, etc. Put the letters in the proper order to spell the secret word! (i) $2x = 6$; (ii) $16 – x = 8$; (iii) $x/3 = 3$; (iv) $7 + (x/6) = 10$.

This class of problems may be enjoyable for some students, but not because of the mathematics. The difficulty and novelty of the problem comes not from the mathematics, but from finding the secret word. Mathematically, this is equivalent to a set of exercises on solving linear equations. Students who can solve these kinds of problems using standard methods will solve the puzzle quickly, and those who can't, will either be stuck, guess the word first and fill in the solutions, or have to be given directions on solving equations. Little mathematical conversation occurs.

Problem 5. The price of a color printer is reduced by 30% of its original price. When it still does not sell, its price is reduced by 20% of the reduced price. The sales person informs you that there has been a total reduction of 50%. Is the sales person using percentages properly? If not, what is the actual percent reduction from the original price?

The problem appears to have room for exploration, but basically has a single solution path. When examined, #5 is a rather constrained problem. The sales person is completely wrong and the erroneous argument is easy to understand. While there are two standard mathematical approaches to the problem, there is little room for student thinking. There is no way into the problem if you don't know an official approach. Either the student understands how to do the problem or not.

Problem 6. Take any three consecutive integers. Multiply the first and last number and compare it to the square of the middle number. Use algebra to explain why this always happens.

This problem might typically be given to a first-year algebra class with experience multiplying binomials, and it is somewhat richer than #4 and #5. For one thing, there is an aspect of exploration and a couple of ways of modeling the situation with algebra. However, the actual exploration is almost trivial, because students see the difference-of-one pattern right away. Then students turn immediately (as asked) to converting the two terms into algebraic expressions. This tends to be either very difficult or very easy for students.

There are three typical ways to write the expressions (depending on whether you think the first number is N, N-1 or sometimes N+1), and there is not much of interest in the problem short of a complete solution. The main richness of the problem is in comparing the two or three ways to represent the problem. Unfortunately in this case, students who grasp one such approach will usually be the ones who will see the other similar symbolic approaches. Furthermore, these approaches will usually be found by high-status students who are confident and fast at symbolic manipulation and simplification. There is a further small shortcut for people who have memorized the special product of a sum and difference. <u>A problem that rewards speedy symbolic manipulation and punishes other approaches leads to very uneven learning in the classroom</u>.

How Are Content, Process, and Product Differentiated in Student's Work on a Rich Problem?

When students work on the Difference of Squares problem, most generally start by trying out numerical examples or by using the factorization of the difference of squares although some may start with geometric models. Each student may start by focusing on different mathematical **content**, but then in the process of working with their peers they have opportunities to learn new content and new connections between content areas. For this particular problem, the content covered varies from new experiences in working with numbers, number sense, and order of operations, to algebraic representation, manipulation, and proof and, in some cases includes a geometry connection. In the linear equations example, some students will be finding a relationship between the slope of a line, the table for its graph, and its equation, while others are trying to figure out or verify a relationship for the slopes of perpendicular lines.

Because there are many accessible entry points to a rich problem, each student and each group can choose an approach or **process** to use with the problem. Some will want to stay with data collection and looking for patterns, while others will be anxious to generalize and use algebra. Visual learners may ask for graph paper, while others will use calculators. Some may try more specific examples to justify their findings, some may use algebra or geometry, others may explain verbally.

One of the beauties of rich problems is that every group gets mathematical results, and usually different groups come up with different **products**. Results from the first problem we discussed can be summarized in a whole class discussion and comparison of different methods, while the second and third examples are great problems for group posters in which each group can show its results. For the Difference of Squares problem, some groups will make lists of examples, look for patterns, and come up with verbal descriptions of their patterns, while others may generalize with algebraic expressions through which they can verify their conjectures. Some may use sums and products of even and odd numbers to justify their reasoning. When each group presents its work on a poster, the posters will all be different. The students who have solved the problem numerically will have had enough concrete experience to be prepared to see their ideas generalized using algebra, and those who focused on a more abstract approach can see their formulas come alive in the numerical patterns.

Conclusion: Challenges In Using Rich Problems

We have seen this approach used effectively in a wide variety of classrooms, some with a few English language learners and some with all students speaking little English. It has worked in classes that include both accelerated students and students who are several years behind grade level. As an approximation, based on our experience, we think it is suitable for the middle 95% of students.

Using rich problems effectively in the classroom requires effective facilitation on the part of the teacher. This is a different approach to teaching from what most teachers have experienced, and it requires time and practice to learn. The teachers we worked with whose students are having some success, have had the opportunity to work together preparing and observing each others lessons during a full year. We document these experiences further in another paper (Hsu et al, in preparation).

A key skill for teachers to learn is how to support groups that are struggling and ones that finish early. A teacher needs to prepare before class what we call “pocket questions.” These must include questions that provide suitable extensions for faster groups and ones that provide enough guidance to get started in a reasonable direction for groups that are getting nowhere or are going up a blind alley. This kind of guidance may be similar to the scaffolding found in some worksheets that spell out subproblems or enumerate smaller steps.

Bringing closure to a group activity so that the point of doing the activity becomes clear to all is another important task for teachers. Often, the purpose of the activity will be to deepen students’ understanding of a key concept. A teacher needs skill in facilitating such whole class discussions and in designing written followup to the lesson so that solid learning takes place. Sometimes this can be done during student presentations, where the teacher calls attention to common threads and connections between different work. Other times, this might consist of a closing statement that restates and packages the important collective results and notes unresolved issues for future study.

Another skill that teachers must learn is how to build a community in the classroom so that students show respect for each other’s views, value working together, and begin to learn the skills that they need to work well in a group. This issue has been examined in other work, for instance the works cited in Footnote 1. Finally, teachers need to develop ‘taste’ so they can recognize rich problems when they see them and are not deceived by ones that only appear to be rich.

As math teachers we can work together with our colleagues to build a library of problems dealing with appropriate content for each course. Clearly it is not always possible to collect sets of problems to cover all the content that may be required for a particular course. But even in the cases that no problem can be found, there are lessons that can begin with rich exploratory investigations that set the stage for students to generate their own questions, and when students have the questions, they are much more interested in learning about some of the methods we are trying to teach.

Acknowledgments

This material is based in part upon work supported by the National Science Foundation under Grants No. 0226972 and 0347784. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Alper, L., Fendel, D. M., Fraser, S., & Resek, D. (2003a). *Interactive Mathematics Program: Year 1*. Emeryville, CA: Key Curriculum Press.

Alper, L., Fendel, D. M., Fraser, S., & Resek, D. (2003b). *Interactive Mathematics Program: Year 2*. Emeryville, CA: Key Curriculum Press.

Clarke, D. (1997). *Constructive Assessment in Mathematics: Practical Steps for Classroom Teachers*. Emeryville, CA: Key College Publishing.

Cohen, E. G. (1994). *Designing Groupwork: Strategies for the Heterogeneous Classroom*. New York, NY: Teachers College Press.

Dietiker, L., Kysh, J., Sallee, T., Hoey, B. (2005). *Algebra Connections.* Sacramento, CA: CPM Educational Program.

Hsu, E., Kysh, J., & Resek, D. (in preparation). Helping Teachers To Use Rich Problems.

Hsu, E., Murphy, T. J., & Treisman, U. (2007). Supporting High Minority Achievement in Introductory Collegiate Mathematics Courses: The Emerging Scholars Program Turns 30. In M. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Practice in Undergraduate Mathematics Education*: Mathematical Association of America.

Sallee, T., Kysh, J., Kasimatis, E., Hoey, B. (1991) *College Preparatory Mathematics: Course 1,* Sacramento, CA, CPM Educational Program.

Tomlinson, C. A. (1999). *The Differentiated Classroom: Responding To The Needs Of All Learners.* Alexandria, VA: Association for Supervision and Curriculum Development.

Webb, N. M., Farivar, S. H., & Mastergeorge, A. M. (2001). *Productive Helping in Cooperative Groups* (No. CSE Technical Report No. 555). Los Angeles, CA: National Center for Research on Evaluation, Standards and Student Testing, UCLA.

Weissglass, J. (1976). Small Groups: An Alternative to the Lecture Method. *Two Year College Mathematics Journal (now The College Mathematics Journal), 7*, 15--20.

Figure 1. A growth pattern problem. Find the next figures in this pattern and a formula for the number of squares in the Nth figure.